

EXACT RINGS AND SEMIRINGS

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ABSTRACT. We introduce and study an abstract class of semirings, which we call exact semirings, defined by a Hahn-Banach-type separation property on modules. Our motivation comes from the tropical semiring, and in particular a desire to understand the often surprising extent to which it behaves like a field. The definition of exactness abstracts an elementary property of fields and the tropical semiring, which we believe is fundamental to explaining this similarity. The class of exact semirings turns out to include many other important examples of both rings (proper quotients of principal ideal domains, matrix rings and finite group rings over these and over fields), and semirings (the Boolean semiring, generalisations of the tropical semiring, matrix semirings and group semirings over these).

1. INTRODUCTION

A *semiring* is an algebraic structure satisfying the usual axioms for a (not necessarily commutative) ring, but without the requirement that addition be invertible. Aside from rings, well-studied examples include the *Boolean semiring* and the *tropical semiring*. The latter is the algebraic structure formed by the real numbers (sometimes considered with $-\infty$ adjoined) under the operations of addition and maximum. It has a huge array of applications in areas including enumerative algebraic geometry and discrete event systems, and has been independently rediscovered many times by people working in these areas.

A lack of additive inverses is clearly a radical departure from the definition of a field, and *a priori* one would not expect linear algebra over any non-ring semiring (even a semifield) to behave much like classical linear algebra over a field. For example, as a direct consequence of the non-invertibility of addition, invertible tropical matrices are extremely sparse [1, Theorem 1.1.3] and have little to do with the structure of matrices in general. There is also no single well-behaved notion of *rank* for tropical matrices, partly as a consequence of which even submodules of free modules can be exceedingly

2010 *Mathematics Subject Classification.* 14T05, 16Y60, 20M10.

Key words and phrases. rings; semirings; Hahn-Banach separation; tropical semiring; Green's relations.

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complex objects. However, as one delves deeper into the theory, one is often struck by surprising large-scale similarities between the tropical and the field case.

A case in point is Green's \mathcal{D} relation (a key tool for capturing the structure of left and right ideals in semigroup theory [7, 9]) for matrices. It is folklore ([12, Theorem 1.4] and [11, Lemma 2.1]) that two matrices (square and of the same size) over a field are \mathcal{D} -related if and only if they have the same rank. Since the isomorphism type of a vector space is uniquely determined by its rank, another way of saying this is that they are \mathcal{D} -related if and only if they have isomorphic column (or equivalently, row) spaces. In [8], Hollings and the third author showed that two matrices over the tropical semiring are also \mathcal{D} -related if and only if they have (tropical linearly) isomorphic column spaces.

A key ingredient in [8] was a lemma establishing a kind of elementary Hahn-Banach separation property, which we shall here call *exactness*. This property, which has a number of equivalent formulations (see Section 3 below), arises in the tropical case from the phenomenon of *tropical matrix duality* [3, 5, 8], and also holds (for completely different and much more elementary reasons) in fields. It has a number of interesting consequences, and indeed we believe it may explain much of the apparent commonality between these ostensibly quite different structures.

From an algebraic perspective, one is drawn to ask whether other important semirings (including rings) are exact in this sense, and hence whether they can be expected to share that behaviour which is common to fields and the tropical semiring. If so, it would be natural to develop an abstract algebraic theory of such semirings. In fact the class of exact semirings with this property does indeed turn out to encompass a number of important semirings. These include the Boolean semiring, proper quotients of principal ideal domains (such as $\mathbb{Z}/n\mathbb{Z}$ for all non-zero $n \in \mathbb{Z}$) and, more generally, self-injective rings. It also includes matrix semirings over exact semirings, and finite (and in some cases infinite) group semirings over exact semirings. On the other hand, many important rings do *not* have this property: indeed we shall see that an integral domain is exact only if it is a field, so \mathbb{Z} , for example, is *not* exact.

In addition to this introduction, this article comprises seven sections. Section 2 recaps the definitions of semirings and various associated concepts. Section 3 introduces exact semirings, proves the equivalence of a number of different characterizations, and establishes some basic properties. Section 4 considers what rings have the exactness property, showing that a proper quotient of a principal ideal domain is always exact, but an integral domain cannot be exact unless it is a field. Section 5 shows that the class of exact semirings is closed under taking matrix semirings; as a consequence we also deduce that a finite group semiring over an exact semiring is exact. Section 6 considers a class of semirings which are in a certain sense *anti-isomorphic* to themselves, showing that these semirings are also exact: these include the Boolean semiring, the tropical semiring and various important generalisations thereof. Section 7 explores a consequence of exactness, generalising a result of [8] to show that there is a straightforward characterisation of

Green's \mathcal{D} relation for matrices with entries in an exact semiring. Finally, Section 8 briefly discusses some questions arising from the results in the preceding sections.

2. PRELIMINARIES

A *semiring* is a commutative semigroup $(S, +)$ with an associative (but not necessarily commutative) multiplication $S \times S \rightarrow S$ that distributes over addition from both sides. In the literature it is generally assumed, see [6] for instance, that a semiring has an additive identity $0 \in S$ that is an absorbing element for multiplication and a multiplicative identity $1 \in S$. However we shall only require our semirings to satisfy the following (weaker) condition: for every non-empty finite subset $L \subseteq S$ there are *local identities* $0_L, 1_L \in S$ with $a + 0_L b = 1_L a = a$ and $a + b 0_L = a 1_L = a$ for all $a, b \in L$. This condition is an adaptation of the 'local zeroes' condition in [8]. It allows us to consider (semirings derived from) the finitary tropical semiring (see Section 6), which is of considerable importance in tropical algebraic geometry.

We write $S^{m \times n}$ for the additive semigroup of m row, n column matrices with entries in S , where $m, n \in \mathbb{N}$. Matrix multiplication behaves in the usual ways: where defined it is associative and distributes over matrix addition. In particular $S^{n \times n}$ is a semiring, called the *matrix semiring* $M_n(S)$, for each $n \in \mathbb{N}$. Note that local identities in $M_n(S)$ can be defined in terms of local identities in S , since the set of entries of a finite set of matrices is a finite subset of S . These local identities are analogues of the standard zero and identity matrices.

Each matrix $A \in S^{m \times n}$ has an associated *row space*

$$\text{Row}(A) = \{x \in S^{1 \times n} : x = uA \text{ for some } u \in S^{1 \times m}\} \quad (2.1)$$

and an associated *column space*

$$\text{Col}(A) = \{y \in S^{m \times 1} : y = Av \text{ for some } v \in S^{n \times 1}\}. \quad (2.2)$$

The local identities in S ensure that $\text{Row}(A)$ actually contains the rows of A , and similarly that $\text{Col}(A)$ contains the columns of A . Notice that the row (column) space of A is closed under addition and left (right) multiplication by 1×1 matrices (which are nothing but elements of S). This suggests that the sets $\text{Row}(A)$ and $\text{Col}(A)$ have 'module' structures under the action of S . We now make this statement precise.

A *left S -module* X is a commutative semigroup $(X, +)$ with an associative left action $S \times X \rightarrow X$ that distributes over addition in X and S , subject to the requirement that for every non-empty finite $K \subseteq X$ there are left local identities $0_K, 1_K \in S$ with $x + 0_K y = 1_K x = x$ for all $x, y \in K$. Right S -modules are similarly defined. In [6] modules are called 'semimodules' and our local identity conditions are replaced by analogous conditions for the (global) identities in S . If S is a (unital) ring then the present definition describes S -modules in the usual sense. In particular, if S is a field then an S -module is just a vector space.

It is clear that S itself is both a left and a right S -module. We consider $S^{1 \times n}$ and row spaces of matrices to be left S -modules, with the obvious left

action, and we consider $S^{m \times 1}$ and column spaces of matrices to be right S -modules, again with the obvious right action.

A function $\phi: X \rightarrow Y$ between left S -modules X and Y is called *left linear* if $\phi(ax + by) = a(\phi x) + b(\phi y)$ for all $x, y \in X$ and all $a, b \in S$. If a left linear (respectively *right linear*; defined in the obvious way) function is bijective then the inverse function is automatically left (right) linear. Such a function $X \rightarrow Y$ is called an *isomorphism* and we write $X \cong Y$. If a linear function is injective then we call it an *embedding*.

The set $\text{Col}(A)^*$ of all right linear functions from the column space of a matrix $A \in S^{m \times n}$ to S is a left S -module. The sum of two functions $\phi, \psi: \text{Col}(A) \rightarrow S$ is given by $(\phi + \psi)y = \phi y + \psi y$ for all $y \in \text{Col}(A)$ and the left action of S on $\phi: \text{Col}(A) \rightarrow S$ is given by $(a\phi)y = a(\phi y)$ for all $a \in S$ and all $y \in \text{Col}(A)$. Left local identities for a finite non-empty $L \subseteq \text{Col}(A)^*$ are given by $0_{L'}$ and $1_{L'}$ where $L' = \{\phi A : \phi \in L\}$, and where $\phi A \in S^{1 \times n}$ is the result of applying ϕ to the columns of A . Similarly the set $\text{Row}(A)^*$ of left linear functions $\text{Row}(A) \rightarrow S$ is a right S -module.

3. EXACT SEMIRINGS

In this section we give the definition of exactness for semirings and we characterise it in ways that are familiar from classical linear algebra over fields. In particular we show that exactness is equivalent to the property of linear functions on row and column spaces extending to linear functions on the appropriate containing modules.

Definition 3.1. *A semiring S is exact if the following conditions hold for all $A \in S^{m \times n}$.*

- (E1) *If $x \in S^{1 \times n} \setminus \text{Row}(A)$ then there are $v, v' \in S^{n \times 1}$ with $Av = Av'$ but $xv \neq xv'$.*
- (E2) *If $y \in S^{m \times 1} \setminus \text{Col}(A)$ then there are $u, u' \in S^{1 \times m}$ with $uA = u'A$ but $uy \neq u'y$.*

Note that if S is commutative then (E1) and (E2) are equivalent because the transpose of any matrix product is the reverse product of the transposes of the factor matrices.

Definition 3.1 motivates a further two definitions. The *kernel* of a set $X \subseteq S^{1 \times n}$ of row vectors is the *right congruence* (equivalence relation that is compatible with addition and the right action of S)

$$\text{Ker}(X) = \{(v, v') \in S^{n \times 1} \times S^{n \times 1} : xv = xv' \text{ for all } x \in X\} \quad (3.1)$$

on $S^{n \times 1}$. In the case that X is the row space of a matrix $A \in S^{m \times n}$, the kernel of X is simply the (set-theoretic) kernel of the surjective right linear function $S^{n \times 1} \rightarrow \text{Col}(A)$ given by $v \mapsto Av$. This observation makes it clear that $S^{n \times 1} / \text{Ker Row}(A) \cong \text{Col}(A)$ as right S -modules. The kernel of a set of column vectors is a similarly defined left congruence on $S^{1 \times m}$, and we have that $S^{1 \times m} / \text{Ker Col}(A) \cong \text{Row}(A)$ as left S -modules.

From the form of (3.1) it is clear that if $X \subseteq Y \subseteq S^{1 \times n}$ then $\text{Ker}(Y) \subseteq \text{Ker}(X)$. In particular if $A \in S^{m \times n}$ and $B \in S^{p \times n}$ with $\text{Row}(A) \subseteq \text{Row}(B)$ then $\text{Ker Row}(B) \subseteq \text{Ker Row}(A)$. That is, $\text{Ker}(-)$ is inclusion-reversing for row (and similarly column) spaces of matrices. The following theorem tells

us that S is exact if and only $\text{Ker}(-)$ is an order anti-embedding (with respect to inclusion) for row and column spaces.

Theorem 3.2. *Let S be a semiring. Then S is exact if and only if the following conditions hold for all $A \in S^{m \times n}$.*

(F1) *If $B \in S^{p \times n}$ with $\text{Ker Row}(A) \subseteq \text{Ker Row}(B)$ then $\text{Row}(B) \subseteq \text{Row}(A)$.*

(F2) *If $B \in S^{m \times q}$ with $\text{Ker Col}(A) \subseteq \text{Ker Col}(B)$ then $\text{Col}(B) \subseteq \text{Col}(A)$.*

Proof. We show the equivalence of (E1) and (F1) for all $A \in S^{m \times n}$, the equivalence of (E2) and (F2) being dual. First suppose that (E1) holds for $A \in S^{m \times n}$. To show that (F1) holds for A let $B \in S^{p \times n}$ with $\text{Ker Row}(A) \subseteq \text{Ker Row}(B)$ and let $x \in \text{Row}(B)$. Then $xv = xv'$ for all $(v, v') \in \text{Ker Row}(A)$, and as such $x \in \text{Row}(A)$ by the contrapositive of (E1). Hence $\text{Row}(B) \subseteq \text{Row}(A)$.

Now suppose that (F1) holds for $A \in S^{m \times n}$. To show that (E1) holds for A let $x \in S^{1 \times n} \setminus \text{Row}(A)$. The matrix

$$B = \begin{bmatrix} A \\ x \end{bmatrix} \in S^{(m+1) \times n} \quad (3.2)$$

satisfies $\text{Row}(A) \subseteq \text{Row}(B)$, so $\text{Ker Row}(B) \subseteq \text{Ker Row}(A)$ because $\text{Ker}(-)$ is inclusion-reversing for row spaces. By construction $\text{Row}(B) \not\subseteq \text{Row}(A)$, so $\text{Ker Row}(B) \subset \text{Ker Row}(A)$ by the contrapositive of (F1). Therefore there is some $(v, v') \in \text{Ker Row}(A)$ with $Bv \neq Bv'$, and hence with $xv \neq xv'$. That is, (E1) holds for A . \square

Given a matrix $A \in S^{m \times n}$ and any $x \in \text{Row}(A)$ we can define a right linear function $\text{Col}(A) \rightarrow S$ by $Av \mapsto xv$. This function is well-defined because x can be written as uA for some $u \in S^{1 \times m}$, so $x \mapsto (Av \mapsto xv)$ is a well-defined function $\text{Row}(A) \rightarrow \text{Col}(A)^*$. This (outer) function is injective, since if $xv = x'v$ for all $v \in S^{n \times 1}$ then we can use local identities to deduce that $x = x'$, and is left linear, so is an embedding of left S -modules. Similarly there is an embedding $\text{Col}(A) \rightarrow \text{Row}(A)^*$ of right S -modules given by $y \mapsto (uA \mapsto uy)$. The following theorem characterises exactness in terms of the surjectivity of these embeddings.

Theorem 3.3. *Let S be a semiring. Then S is exact if and only if the following conditions hold for all $A \in S^{m \times n}$.*

(G1) *The left embedding $\text{Row}(A) \rightarrow \text{Col}(A)^*$ given by $x \mapsto (Av \mapsto xv)$ is surjective.*

(G2) *The right embedding $\text{Col}(A) \rightarrow \text{Row}(A)^*$ given by $y \mapsto (uA \mapsto uy)$ is surjective.*

Proof. First suppose that (E1) holds for $A \in S^{m \times n}$. Let $\phi \in \text{Col}(A)^*$ and suppose that $\phi A \notin \text{Row}(A)$, where $\phi A \in S^{1 \times n}$ is the result of applying ϕ to the columns of A . Then by (E1) there are $v, v' \in S^{n \times 1}$ with $Av = Av'$ and $(\phi A)v \neq (\phi A)v'$. However, right linearity of ϕ gives $\phi(Av) = (\phi A)v$ and $\phi(Av') = (\phi A)v'$, so we have $(\phi A)v = (\phi A)v'$. This contradicts $(\phi A)v \neq (\phi A)v'$, and as such we must actually have $\phi A \in \text{Row}(A)$. Therefore ϕ is the image of ϕA under the given function $\text{Row}(A) \rightarrow \text{Col}(A)^*$. Hence (G1) holds for A .

Now suppose that (G1) holds for $A \in S^{m \times n}$. We show that (F1) holds for A . Let $B \in S^{p \times n}$ with $\text{Ker Row}(A) \subseteq \text{Ker Row}(B)$ and let $x \in \text{Row}(B)$. The right linear function $\text{Col}(A) \rightarrow S$ given by $Av \mapsto xv$ is then well-defined: if $Av = Av'$ then $(v, v') \in \text{Ker Row}(A) \subseteq \text{Ker Row}(B)$, so $xv = uBv = uBv' = xv'$ where $x = uB$ for some $u \in S^{1 \times p}$. By (G1) this function $\text{Col}(A) \rightarrow S$ is given by $Av \mapsto x'v$ for some $x' \in \text{Row}(A)$, and as such $xv = x'v$ for all $v \in S^{n \times 1}$. We can then use local identities to deduce that $x = x' \in \text{Row}(A)$, and therefore $\text{Row}(B) \subseteq \text{Row}(A)$. Hence (F1) holds for A .

As above, a similar argument shows that (G2) is equivalent, via (F2), to (E2) for all $A \in S^{m \times n}$. \square

Theorem 3.3 tells us that if a semiring S is exact then $\text{Row}(A) \cong \text{Col}(A)^*$ as left S -modules and $\text{Col}(A) \cong \text{Row}(A)^*$ as right S -modules for all $A \in S^{m \times n}$. It also tells us that every right linear function $\text{Col}(A) \rightarrow S$ can be written as an inner product, and so in particular every right linear function $\text{Col}(A) \rightarrow S$ extends to a right linear function $S^{m \times 1} \rightarrow S$. Similarly every left linear function $\text{Row}(A) \rightarrow S$ extends to a left linear function $S^{1 \times n} \rightarrow S$. In fact, taken together these properties turn out to be yet another characterisation of exactness.

Theorem 3.4. *Let S be a semiring. Then S is exact if and only if the following conditions hold for all $A \in S^{m \times n}$.*

- (H1) *Every right linear function $\text{Col}(A) \rightarrow S$ extends to a right linear function $S^{m \times 1} \rightarrow S$.*
- (H2) *Every left linear function $\text{Row}(A) \rightarrow S$ extends to a left linear function $S^{1 \times n} \rightarrow S$.*

Proof. We have already observed that (H1) follows from (G1), so suppose that (H1) holds for $A \in S^{m \times n}$ and let $\phi \in \text{Col}(A)^*$. Now let $I \in M_m(S)$ be an $m \times m$ identity matrix local to A , with $IA = A$. Recall that ϕA is the result of applying ϕ to the columns of A . Since ϕ extends to $S^{m \times 1}$ we may also apply ϕ to the columns of I , giving ϕI . Right linearity of ϕ then gives $\phi(IAv) = (\phi I)Av$ for all $Av \in \text{Col}(A)$. Therefore $\phi(Av) = (\phi I)Av$ for all $Av \in \text{Col}(A)$ because $IA = A$, and as such ϕ is the image of $(\phi I)A$ under the function $\text{Row}(A) \rightarrow \text{Col}(A)^*$ given in (G1). Hence (G1) holds for A .

Similarly (H2) and (G2) are equivalent for all $A \in S^{m \times n}$. \square

Recall that a semiring S is *self-injective* if every linear function $X \rightarrow S$, for X an arbitrary S -module, factors through every embedding of X into any S -module. It is clear from Theorem 3.4 that exactness of S is nothing but self-injectivity restricted to the embeddings $\text{Col}(A) \subseteq S^{m \times 1}$ and $\text{Row}(A) \subseteq S^{1 \times n}$. Thus every self-injective semiring is exact.

4. ORTHOGONAL COMPLEMENTS AND EXACT RINGS

Throughout this section R will denote a *ring* (with unity, but not necessarily commutative), which for our purposes may be thought of as a semiring which has a global zero and identity (denoted 0 and 1) and which forms a group under addition (the inverse of $a \in R$ being denoted $-a$).

The *orthogonal complement* of a set $X \subseteq R^{1 \times n}$ is the right R -module

$$X^\perp = \{v \in R^{n \times 1} : xv = 0 \text{ for all } x \in X\}. \quad (4.1)$$

Similarly the orthogonal complement of a set $Y \subseteq R^{m \times 1}$ is the left R -module

$$Y^\perp = \{u \in R^{1 \times m} : uy = 0 \text{ for all } y \in Y\}. \quad (4.2)$$

In the case $n = m = 1$ the notation ' $^\perp$ ' is ambiguous, but the appropriate definition of orthogonal complement will always be clear from context. If in this case R is commutative then the two notions of orthogonal complement coincide and X^\perp is just the *annihilator* $\text{Ann}(X)$ of $X \subseteq R$.

If $A \in R^{m \times n}$ then $\text{Row}(A)^\perp$ is the kernel of the surjective right linear function $R^{n \times 1} \rightarrow \text{Col}(A)$ given by $v \mapsto Av$. Therefore $R^{n \times 1} / \text{Row}(A)^\perp \cong \text{Col}(A)$ as right R -modules. Similarly $R^{1 \times m} / \text{Col}(A)^\perp \cong \text{Row}(A)$ as left R -modules. As with the kernels of row and column spaces, taking orthogonal complements reverses inclusions. We also have $\text{Row}(A) \subseteq \text{Row}(A)^{\perp\perp}$ and $\text{Col}(A) \subseteq \text{Col}(A)^{\perp\perp}$ for all $A \in R^{m \times n}$, with (by the following theorem) equality if and only if R is exact.

Theorem 4.1. *Let R be a ring. Then R is exact if and only if $\text{Row}(A)^{\perp\perp} = \text{Row}(A)$ and $\text{Col}(A)^{\perp\perp} = \text{Col}(A)$ for all $A \in R^{m \times n}$.*

Proof. Suppose that (E1) holds for $A \in R^{m \times n}$ and let $x \in \text{Row}(A)^{\perp\perp}$. Then $xv = xv'$ for all $(v, v') \in \text{Ker Row}(A)$ because $v - v' \in \text{Row}(A)^\perp$. Therefore $x \in \text{Row}(A)$ by the contrapositive of (E1), and as such $\text{Row}(A)^{\perp\perp} \subseteq \text{Row}(A)$. Hence $\text{Row}(A)^{\perp\perp} = \text{Row}(A)$ because, as noted above, we always have $\text{Row}(A) \subseteq \text{Row}(A)^{\perp\perp}$.

Now suppose that $\text{Row}(A)^{\perp\perp} = \text{Row}(A)$ for $A \in R^{m \times n}$ and let $B \in R^{p \times n}$ with $\text{Ker Row}(A) \subseteq \text{Ker Row}(B)$. Then $\text{Row}(A)^\perp \subseteq \text{Row}(B)^\perp$, so

$$\text{Row}(B) \subseteq \text{Row}(B)^{\perp\perp} \subseteq \text{Row}(A)^{\perp\perp} = \text{Row}(A) \quad (4.3)$$

because taking orthogonal complements reverses inclusions. Hence (F1) holds for A .

A similar argument shows that (E2) implies $\text{Col}(A)^{\perp\perp} = \text{Col}(A)$, and that this in turn implies (F2), for all $A \in R^{m \times n}$. \square

Recall that a ring is called an *integral domain* if it is commutative, it has no zero divisors and $0 \neq 1$. The following proposition tells us that every exact integral domain must be a field, so in particular the prototypical integral domain \mathbb{Z} is not exact.

Proposition 4.2. *If R is an exact commutative ring then every non-zero element of R is a zero divisor or a unit.*

Proof. Let $a \in R \setminus \{0\}$. If a is a zero divisor then we are done, so suppose that $ab \neq 0$ for all $b \in R \setminus \{0\}$. Then the function $\phi \in (aR)^*$ given by $\phi(av) = v$ is well-defined. Hence $1 = \phi a = ua$ for some $u \in R$ by (G1). \square

We recall that a *principal ideal domain* is an integral domain in which every ideal is generated by a single element. All fields, \mathbb{Z} and $K[t]$ for K a field are principal ideal domains, where $K[t]$ denotes the ring of polynomials in t with coefficients from K . For the remainder of this section P will be a principal ideal domain and R will be the quotient of P by (the ideal generated by) a fixed element $r \in P$. We may assume that for every $A \in R^{m \times n}$ there are invertible matrices $M \in M_m(R)$ and $N \in M_n(R)$ with MAN diagonal [4, Theorem 10.4].

Theorem 4.3. *If $R = P/rP$ for P a principal ideal domain and $r \in P$ then $\text{Row}(A) \cong \text{Col}(A)$ for all $A \in R^{m \times n}$.*

Proof. Adding rows and columns of zeroes to A does not (up to isomorphism at least) change its row space or column space, so we may assume that A is square and (by [4, Theorem 10.4]) that there are invertible matrices M and N with MAN diagonal. Then

$$\begin{aligned} ANM^{-T} &= M^{-1}MANM^{-T} \\ &= M^{-1}(MAN)^T M^{-T} \\ &= M^{-1}N^T A^T \\ &= (ANM^{-T})^T \end{aligned} \tag{4.4}$$

because MAN is symmetric, and as such ANM^{-T} is symmetric.

The required isomorphism $\phi: \text{Row}(A) \rightarrow \text{Col}(A)$ is given by $\phi(uA) = (uANM^{-T})^T$ for all $uA \in \text{Row}(A)$, where the image of ϕ is contained in $\text{Col}(A)$ because ANM^{-T} is symmetric. The inverse of ϕ is given by $\phi^{-1}(Av) = (N^{-T}MAv)^T$ for all $Av \in \text{Col}(A)$. \square

Theorem 4.3 places no restriction on $r \in P$, so it applies to (for instance) \mathbb{Z} and $\mathbb{Z}/n\mathbb{Z}$ for all non-zero $n \in \mathbb{Z}$. The following results also apply to all $\mathbb{Z}/n\mathbb{Z}$ with $n \neq 0$. They do not, however, apply to \mathbb{Z} .

Theorem 4.4. *Let $R = P/rP$ for P a principal ideal domain and $r \in P \setminus \{0\}$. If $A \in R^{m \times n}$ then there is some $B \in R^{n \times n}$ with $\text{Row}(A)^\perp = \text{Col}(B)$ and $\text{Col}(B)^\perp = \text{Row}(A)$.*

Proof. We will first show that the result holds for all $A \in R^{1 \times 1}$. That is, we will show that for all $A \in R$ there is some $B \in R$ with $\text{Ann}(RA) = BR$ and $\text{Ann}(BR) = RA$.

Write $A = a' + rP$ for some $a' \in P$ and take $a \in P$ to generate the ideal $Pa' + rP$. That is, take a to be the greatest common divisor of a' and r in P . Then $r = ba$ for some $b \in P$ and $RA = R(a + rP)$. If $a = 0$ or $b = 0$ then $r = 0$, so we must have $a \neq 0$ and $b \neq 0$. Now write $B = b + rP$. Then we have $BR \subseteq \text{Ann}(RA)$ because $ba \in rP$. If $c + rP \in \text{Ann}(RA)$ then $ca = dr = dba$ for some $d \in P$, so $c = db$ because P is an integral domain and $a \neq 0$. Therefore $c + rP \in BR$, and as such $\text{Ann}(RA) = BR$. Similarly $\text{Ann}(BR) = RA$. Note that if $A = 0 + rP$ then we can take $B = 1 + rP$.

This result extends to diagonal matrices $A \in R^{m \times n}$. Without loss of generality (by adding or removing rows of zeroes as necessary) we may assume that A is square diagonal. To form $B \in R^{n \times n}$ replace each diagonal entry $a \in R$ of A by an element $b \in R$ using the above procedure. This will ensure that $\text{Row}(A)^\perp = \text{Col}(B)$ and $\text{Col}(B)^\perp = \text{Row}(A)$.

We can now prove the theorem for an arbitrary matrix $A \in R^{m \times n}$. By [4, Theorem 10.4] there are invertible matrices $M \in M_m(R)$ and $N \in M_n(R)$ with MAN diagonal, so by the above result there is a matrix $B \in R^{n \times n}$ with $\text{Row}(MAN)^\perp = \text{Col}(B)$ and $\text{Col}(B)^\perp = \text{Row}(MAN)$. Invertibility of M and N then gives

$$\text{Row}(A)^\perp = \text{Row}(MA)^\perp = \text{Col}(NB) \tag{4.5}$$

and

$$\text{Col}(NB)^\perp = \text{Row}(MA) = \text{Row}(A). \quad (4.6)$$

□

Corollary 4.5. *If $R = P/rP$ for P a principal ideal domain and $r \in P \setminus \{0\}$ then R is exact and*

$$\text{Row}(A)^\perp \cong R^{1 \times n} / \text{Row}(A) \quad (4.7)$$

for all $A \in R^{m \times n}$.

Proof. By Theorem 4.4 there is some $B \in R^{n \times n}$ with $\text{Row}(A)^\perp = \text{Col}(B)$ and $\text{Col}(B)^\perp = \text{Row}(A)$. Therefore $\text{Row}(A)^{\perp\perp} = \text{Col}(B)^\perp = \text{Row}(A)$. We could verify, using a result dual to Theorem 4.4, that $\text{Col}(A)^{\perp\perp} = \text{Col}(A)$ too, but there is no need since R is commutative. Hence R is exact by Theorem 4.1.

For the second claim, by Theorem 4.3 we have $\text{Row}(A)^\perp = \text{Col}(B) \cong \text{Row}(B)$. Therefore $\text{Row}(A)^\perp \cong R^{1 \times n} / \text{Col}(B)^\perp = R^{1 \times n} / \text{Row}(A)$. □

If $r \neq 0$ then R is known to be self-injective [13, Theorem 4.35], so R is exact by Theorem 3.4. Nevertheless, Theorem 4.4 and Corollary 4.5 are included here as the proof of Theorem 4.4 describes a procedure to compute the orthogonal complement of the row space of a matrix with entries in R .

5. MATRIX SEMIRINGS AND GROUP SEMIRINGS

A subset T of a semiring S will be called a *subsemiring* of S if it is closed under addition and multiplication, and if for every non-empty finite $L \subseteq S$ there are $0_L, 1_L \in T$ with $a + 0_L b = 1_L a = a$ and $a + b 0_L = a 1_L = a$ for all $a, b \in L$. That is, for T to be a subsemiring of S we must be able to choose local identities that lie in T , not just in S (as the present definition of a semiring requires), for every non-empty finite subset of S . This is by analogy with the standard definition [6] of a subsemiring T , which requires T to contain $0, 1 \in S$.

It is clear from the relevant definitions that if T is a subsemiring of S then S is both a left and a right T -module. A subsemiring T of S will be called a *left (right) retract* of S if there is a left (right) T -linear function $S \rightarrow T$ that fixes T pointwise.

Theorem 5.1. *Let S be an exact semiring and T be a (right and left) retract of $M_n(S)$. If $S^{1 \times n}$ and $S^{n \times 1}$ embed into T , as right and left T -modules respectively, then T is exact.*

Proof. To apply Theorem 3.2 (which will establish the exactness of T) we need to verify that (F1) and (F2) hold for all $A \in T^{m \times q}$. We will only show that (F1) holds for all A , the proof that (F2) holds being dual.

Let $A \in T^{m \times q}$. Since $T \subseteq M_n(S)$ we can view A as a matrix with entries in S that is divided into blocks of size $n \times n$. That is, we can view A as an element of $S^{mn \times qn}$. We write $\text{Row}_S(A)$ for the row space of A when considered as a matrix with entries in S and to prevent ambiguity we write $\text{Row}_T(A)$ for the row space of $A \in T^{m \times q}$.

Let $B \in T^{p \times q}$ and suppose that $\text{Ker Row}_T(A) \subseteq \text{Ker Row}_T(B)$. We will show that $\text{Ker Row}_S(A) \subseteq \text{Ker Row}_S(B)$. By assumption there is an injective left T -linear function $\phi: S^{n \times 1} \rightarrow T$. Left T -linearity of ϕ gives

$$A(\phi v) = \begin{bmatrix} \phi(X_1 v) \\ \vdots \\ \phi(X_m v) \end{bmatrix} \quad (5.1)$$

for all $v \in S^{qn \times 1}$, where $X_1, \dots, X_m \in T^{1 \times q}$ are the rows of A and $\phi v \in T^{q \times 1}$ is the result of applying ϕ to the q blocks of v . Therefore if $v, v' \in S^{qn \times 1}$ with $Av = Av'$ then $A(\phi v) = A(\phi v')$ because $X_i v = X_i v'$ for each $1 \leq i \leq m$. That is, if $(v, v') \in \text{Ker Row}_S(A)$ then $(\phi v, \phi v') \in \text{Ker Row}_T(A)$. In fact, as ϕ is injective we have

$$(v, v') \in \text{Ker Row}_S(A) \Leftrightarrow (\phi v, \phi v') \in \text{Ker Row}_T(A) \quad (5.2)$$

for all $v, v' \in S^{qn \times 1}$, and similarly for B . Hence $\text{Ker Row}_S(A) \subseteq \text{Ker Row}_S(B)$ because $\text{Ker Row}_T(A) \subseteq \text{Ker Row}_T(B)$.

Since S is exact, (F1) holds for $A \in S^{mn \times qn}$. Therefore $\text{Row}_S(B) \subseteq \text{Row}_S(A)$ because $\text{Ker Row}_S(A) \subseteq \text{Ker Row}_S(B)$. To verify that (F1) holds for $A \in T^{m \times q}$ we need to show that $\text{Row}_T(B) \subseteq \text{Row}_T(A)$, so let $X \in \text{Row}_T(B) \subseteq T^{1 \times q}$. Then each row $x \in S^{1 \times qn}$ of X is in $\text{Row}_S(B) \subseteq \text{Row}_S(A)$, so each row $x \in S^{1 \times qn}$ of X can be written as $x = uA$ for some $u \in S^{1 \times mn}$. Therefore $X = UA$ for some $U \in S^{m \times mn}$.

By assumption again there is a right T -linear function $\pi: M_n(S) \rightarrow T$ that fixes T pointwise. Right T -linearity of π gives

$$(\pi U)A = [\pi(UY_1) \ \cdots \ \pi(UY_q)], \quad (5.3)$$

where $Y_1, \dots, Y_q \in T^{m \times 1}$ are the columns of A and $\pi U \in T^{1 \times m}$ is the result of applying π to the m blocks of U . Since $UA = X \in T^{1 \times q}$, the q entries of X are $UY_1, \dots, UY_q \in T$. Therefore

$$X = [UY_1 \ \cdots \ UY_q] = (\pi U)A \quad (5.4)$$

because π fixes T pointwise, and as such $X \in \text{Row}_T(A)$. Hence $\text{Row}_T(B) \subseteq \text{Row}_T(A)$, as required for (F1) to hold for $A \in T^{m \times q}$. \square

Theorem 5.1 gives sufficient conditions for a (sub)semiring of matrices to be exact. In particular it tells us that if S is exact then every matrix semiring $M_n(S)$ will be exact too, because $S^{1 \times n}$ and $S^{n \times 1}$ clearly embed into $M_n(S)$.

Corollary 5.2. *If S is an exact semiring then every matrix semiring $M_n(S)$ is exact.*

Now let S be a semiring and $(\Gamma, \cdot, 1)$ be a finite group. The set of functions $\Gamma \rightarrow S$ is a semiring with operations defined by $(f + g)\alpha = f\alpha + g\alpha$ and

$$(fg)\alpha = \sum_{\beta\gamma=\alpha} f\beta \cdot g\gamma = \sum_{\beta \in \Gamma} f\beta \cdot g(\beta^{-1}\alpha) \quad (5.5)$$

for all $f, g: \Gamma \rightarrow S$ and all $\alpha \in \Gamma$. We call this semiring the *group semiring* $S\Gamma$. Local identities for a finite non-empty subset $L \subseteq S\Gamma$ can be defined in terms of $0_{L'}, 1_{L'} \in S$, where $L' = \{f\alpha : f \in L \text{ and } \alpha \in \Gamma\}$. Notice that L' is finite because L and Γ are finite.

Theorem 5.3. *If S is an exact semiring and Γ is a finite group then the group semiring $S\Gamma$ is exact.*

Proof. Let Γ be a finite group with $|\Gamma| = n$ and $\Gamma = \{\alpha_1, \dots, \alpha_n\}$. We begin by showing that $S\Gamma$ can be identified with a subsemiring of $M_n(S)$ via the map sending a function $f \in S\Gamma$ to the matrix

$$F = \begin{bmatrix} f(\alpha_1^{-1}\alpha_1) & \cdots & f(\alpha_1^{-1}\alpha_n) \\ \vdots & \ddots & \vdots \\ f(\alpha_n^{-1}\alpha_1) & \cdots & f(\alpha_n^{-1}\alpha_n) \end{bmatrix}. \quad (5.6)$$

Notice that we can recover f from any row or column of F , so the map sending f to F is injective. If $f, g \in S\Gamma$ with corresponding matrices $F, G \in M_n(S)$ then

$$F + G = \begin{bmatrix} (f+g)(\alpha_1^{-1}\alpha_1) & \cdots & (f+g)(\alpha_1^{-1}\alpha_n) \\ \vdots & \ddots & \vdots \\ (f+g)(\alpha_n^{-1}\alpha_1) & \cdots & (f+g)(\alpha_n^{-1}\alpha_n) \end{bmatrix} \quad (5.7)$$

by the definition of addition in $S\Gamma$, so $F + G$ is the matrix corresponding to $f+g \in S\Gamma$. It therefore remains to show that FG is the matrix corresponding to $fg \in S\Gamma$. For each $1 \leq i, j \leq n$ we have

$$(FG)_{ij} = \sum_{k=1}^n F_{ik} \cdot G_{kj} = \sum_{k=1}^n f(\alpha_i^{-1}\alpha_k) \cdot g(\alpha_k^{-1}\alpha_j) \quad (5.8)$$

by (5.6), so

$$(FG)_{ij} = \sum_{\beta \in \Gamma} f\beta \cdot g(\beta^{-1}\alpha_j) = (fg)(\alpha_i^{-1}\alpha_j) \quad (5.9)$$

for each $1 \leq i, j \leq n$ by (5.5). Hence FG is the matrix corresponding to fg . We can therefore treat $S\Gamma$ as if it were the subsemiring of $n \times n$ matrices of the form (5.6).

Note that $S\Gamma$ is a subsemiring of $M_n(S)$ in the sense defined above because we can take local identities for any subset of matrices that are analogues of the standard zero and identity matrices. Therefore we can take local identities that are of the form (5.6).

We will ultimately use Theorem 5.1 to show that $S\Gamma \subseteq M_n(S)$ is exact, but first we introduce some notation. Let $\rho: M_n(S) \rightarrow S^{1 \times n}$ and $\kappa: M_n(S) \rightarrow S^{n \times 1}$ be the functions that select the first row and first column respectively of a matrix $A \in M_n(S)$. It is clear that ρ is right $M_n(S)$ -linear, so in particular it is right $S\Gamma$ -linear. Similarly κ is left $S\Gamma$ -linear. Now let $\psi: S^{1 \times n} \rightarrow S\Gamma$ be the function that sends $x \in S^{1 \times n}$ to the unique, by (5.6), element of $S\Gamma$ with first row x . Similarly let $\phi: S^{n \times 1} \rightarrow S\Gamma$ be the function that sends $v \in S^{n \times 1}$ to the unique element of $S\Gamma$ with first column v .

If $x \in S^{1 \times n}$ then ψx has first row x , so $\rho\psi x = x$. On the other hand, if $F \in S\Gamma$ then $\psi\rho F = F$ because $\psi\rho F$ and F are in $S\Gamma$ and have the same first row. Therefore ρ restricted to $S\Gamma$ has inverse ψ , and as such ψ is left $S\Gamma$ -linear because ρ is. Similarly κ restricted to $S\Gamma$ has inverse ϕ , so ϕ is right $S\Gamma$ -linear.

We are now ready to apply Theorem 5.1. We first show that $S\Gamma \subseteq M_n(S)$ is a right and a left retract of $M_n(S)$. For the right $S\Gamma$ -linear function

$M_n(S) \rightarrow ST$ take $\psi \circ \rho$ and for the left ST -linear function $M_n(S) \rightarrow ST$ take $\phi \circ \kappa$. As noted above, these functions fix ST pointwise, so ST is both a right and a left retract of $M_n(S)$. For the right ST -module embedding $S^{1 \times n} \rightarrow ST$ take ψ (which is injective by definition) and for the left ST -module embedding $S^{n \times 1} \rightarrow ST$ take ϕ . Theorem 5.1 then ensures that ST is exact. \square

6. TROPICAL, BOOLEAN AND OTHER ANTI-INVOLUTIVE SEMIRINGS

In this section we define what it means for a semiring to be anti-involutive and we show that all such semirings are exact. We also show that the row and column spaces of matrices with entries in an anti-involutive semiring are, in a sense defined below, anti-isomorphic as modules.

Recall that a semiring S is *idempotent* if $a + a = a$ for all $a \in S$. If in this case X is an S -module then it follows that $x + x = x$ for all $x \in X$ too, and as such we can define a partial order on X by setting

$$x \leq y \iff x + y = y \quad (6.1)$$

for all $x, y \in X$. In particular S itself can be partially ordered in this way. Like addition, the partial order on S extends entrywise to matrices (of the same size) with entries in S .

Example 6.1. Let $(\Gamma, \cdot, 1)$ be a group with a total order that is compatible with the group operation, by which we mean that if $\alpha, \beta \in \Gamma$ with $\alpha \leq \beta$ then $\gamma\alpha \leq \gamma\beta$ and $\alpha\gamma \leq \beta\gamma$ for all $\gamma \in \Gamma$. For instance Γ could be $(\mathbf{Z}, +, 0)$, $(\mathbf{Q}, +, 0)$ or $(\mathbf{R}, +, 0)$ with the standard ordering. We can turn Γ into an idempotent semiring by defining

$$\alpha + \beta = \max\{\alpha, \beta\} \quad (6.2)$$

for all $\alpha, \beta \in \Gamma$. Notice that the partial order induced by this operation is just the original total order on Γ . Local identities for a non-empty finite $L \subseteq \Gamma$ are $1_L = 1$ and

$$0_L = \min\{(\min L)(\max L)^{-1}, (\max L)^{-1}(\min L)\}. \quad (6.3)$$

In order to distinguish it from the group Γ we write Γ_{\max} for this semiring.

A function $\phi: X \rightarrow Y$ between left S -modules X and Y , for S an idempotent semiring, will be called *left monotone* if

$$ax \leq x' \implies a(\phi x) \leq \phi x' \quad (6.4)$$

for all $x, x' \in X$ and all $a \in S$. It is clear that if ϕ is left linear (e.g. matrix multiplication on the left) then ϕ is left monotone, but the converse is not true in general. For example, the function $\phi: \mathbb{R}_{\max}^{1 \times 2} \rightarrow \mathbb{R}_{\max}$ given by

$$\phi \begin{bmatrix} x_1 & x_2 \end{bmatrix} = -\max\{-x_1, -x_2\} \quad (6.5)$$

is left monotone but not left linear since

$$\phi \max\left\{\begin{bmatrix} 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}\right\} \neq \max\left\{\phi \begin{bmatrix} 1 & 0 \end{bmatrix}, \phi \begin{bmatrix} 0 & 1 \end{bmatrix}\right\}. \quad (6.6)$$

However, if ϕ is left monotone and has a left monotone inverse then ϕ (and hence also ϕ^{-1}) is left linear, as the following proposition shows.

Proposition 6.2. *Let S be an idempotent semiring and $\phi: X \rightarrow Y$ be a function between left (right) S -modules. Then ϕ is an isomorphism of left (right) S -modules if and only if ϕ is left (right) monotone and has a left (right) monotone inverse.*

Proof. If ϕ is an isomorphism then ϕ and ϕ^{-1} are left linear, so are left monotone. It therefore remains to show the converse. Let $x, y \in X$ and $a, b \in S$. Then $ax \leq ax + by$ and $by \leq ax + by$, so $a(\phi x) \leq \phi(ax + by)$ and $b(\phi y) \leq \phi(ax + by)$ because ϕ is left monotone. Therefore $a(\phi x) + b(\phi y) \leq \phi(ax + by)$. We also have $a(\phi x) \leq a(\phi x) + b(\phi y)$ and $b(\phi y) \leq a(\phi x) + b(\phi y)$, so

$$ax \leq \phi^{-1}(a(\phi x) + b(\phi y)) \quad (6.7)$$

and

$$by \leq \phi^{-1}(a(\phi x) + b(\phi y)) \quad (6.8)$$

because ϕ^{-1} is left monotone. Therefore $\phi(ax + by) \leq a(\phi x) + b(\phi y)$, because ϕ is order-preserving, with $a(\phi x) + b(\phi y) \leq \phi(ax + by)$ from above. Hence ϕ is left linear, and as such ϕ is an isomorphism of left S -modules. \square

In response to Proposition 6.2, a function $\phi: X \rightarrow Y$ from a left S -module X to a right S -module Y will be called *antitone* if

$$ax \leq x' \quad \Rightarrow \quad (\phi x')a \leq \phi x \quad (6.9)$$

for all $x, x' \in X$ and all $a \in S$. Antitone functions are (in particular) order-reversing. If ϕ is antitone and has an antitone inverse $\phi^{-1}: Y \rightarrow X$ then we call ϕ (and also ϕ^{-1}) an *anti-isomorphism* and we say that X and Y are *anti-isomorphic* (as modules). The condition for ϕ^{-1} to be antitone is just the obvious analogue of (6.9):

$$ya \leq y' \quad \Rightarrow \quad a(\phi^{-1}y') \leq \phi^{-1}y \quad (6.10)$$

for all $y, y' \in Y$ and all $a \in S$. Note that by Proposition 6.2 the composition of two anti-isomorphisms is an isomorphism of left (or right, whichever is appropriate) modules.

For the remainder of this section S will be an idempotent semiring with an involutive anti-isomorphism $\overline{}: S \rightarrow S$. We call such semirings *anti-involutive*. Ordinarily, specifying that a function $\phi: S \rightarrow S$ is an anti-isomorphism is ambiguous if S is not commutative: should (6.9) and (6.10) apply as written, or should ϕ and ϕ^{-1} be interchanged? However, the fact that $\overline{}$ is an involution removes this distinction, meaning that $\overline{}$ is antitone in the sense of (6.9) and in the sense of (6.10).

The semirings Γ_{\max} in Example 6.1 are anti-involutive with $\overline{\alpha} = \alpha^{-1}$ for all $\alpha \in \Gamma$. In particular the *finitary tropical* semiring $\mathbb{FT} = \mathbb{R}_{\max}$ is anti-involutive with $\overline{\alpha} = -\alpha$ for all $\alpha \in \mathbb{R}$. Another anti-involutive semiring is the *Boolean* semiring $\mathbb{B} = \{\perp, \top\}$, where addition is given by ‘or’, multiplication by ‘and’ and the involution by ‘not’.

The involution on S extends to matrices with entries in S . Specifically, given $A \in S^{m \times n}$ we define $\overline{A} \in S^{n \times m}$ by $\overline{A}_{ij} = \overline{A_{ji}}$ for all $1 \leq i \leq m$ and all $1 \leq j \leq n$. Proposition 6.4 (below) tells us that $\overline{}$ is an involutive

anti-isomorphism on $M_n(S)$ for all n , so every matrix semiring $M_n(S)$ is anti-involutive. Every group semiring $S\Gamma$ for Γ a finite group is anti-involutive with $\overline{f}\alpha = \overline{f}\alpha^{-1}$ for $f \in S\Gamma$ and all $\alpha \in \Gamma$. This also follows from the identification of $S\Gamma$ as a subsemiring of matrices in the proof of Theorem 5.3.

In fact, if S is *complete* in the sense of order theory (so that sums of arbitrarily many elements are possible, such as in \mathbb{B}) then Proposition 6.4 actually holds for infinite matrices and, by implication, elements of $S\Gamma$ for an infinite group Γ . We will not discuss this idea further here, except to say that the following example can be applied to an arbitrary group Γ .

Example 6.3. A group semiring $\mathbb{B}\Gamma$ can be viewed as the semiring of subsets of Γ , where addition is union and multiplication is given by $AB = \{\alpha\beta : \alpha \in A \text{ and } \beta \in B\}$ for all $A, B \subseteq \Gamma$. Such a semiring is anti-involutive with $\overline{A} = \Gamma \setminus A^{-1}$ for all $A \subseteq \Gamma$, where $A^{-1} = \{\alpha^{-1} : \alpha \in A\}$.

Proposition 6.4. If S is an anti-involutive semiring then

$$MA \leq B \Rightarrow \overline{B}M \leq \overline{A} \quad (6.11)$$

for all $A \in S^{m \times n}$, all $B \in S^{p \times n}$ and all $M \in S^{p \times m}$.

Proof. Suppose that $MA \leq B$. We will show that $(\overline{B}M)_{ji} \leq \overline{A}_{ji}$ for all $1 \leq i \leq m$ and all $1 \leq j \leq n$. Since $MA \leq B$ we have $M_{ki}A_{ij} \leq (MA)_{kj} \leq B_{kj}$ for all $1 \leq k \leq p$, so $\overline{B}_{kj}M_{ki} \leq \overline{A}_{ij}$ for all k because $\overline{\cdot} : S \rightarrow S$ is antitone. Therefore $\overline{B}_{jk}M_{ki} \leq \overline{A}_{ji}$ for all k , and as such

$$(\overline{B}M)_{ji} = \sum_{k=1}^p \overline{B}_{jk}M_{ki} \leq \overline{A}_{ji}. \quad (6.12)$$

Hence $\overline{B}M \leq \overline{A}$. \square

An immediate consequence of Proposition 6.4 is that

$$MA \leq B \Rightarrow \overline{B}M \leq \overline{A} \Rightarrow A\overline{B} \leq \overline{M} \Rightarrow MA \leq B \quad (6.13)$$

for all $A \in S^{m \times n}$, all $B \in S^{p \times n}$ and all $M \in S^{p \times m}$. In the arguments that follow we make frequent use of (6.13). Informally, it allows us to cycle terms in an inequality at the cost of introducing a $\overline{\cdot}$ to the two terms that cross the inequality. We also use the fact that matrix multiplication is order-preserving.

Theorem 6.5. If S is an anti-involutive semiring then $\text{Row}(A)$ and $\text{Col}(A)$ are anti-isomorphic for all $A \in S^{m \times n}$.

Proof. We exhibit an antitone function $\phi : \text{Row}(A) \rightarrow \text{Col}(A)$ that has an antitone inverse $\text{Col}(A) \rightarrow \text{Row}(A)$. Define ϕ by $\phi x = A\overline{x}$ for all $x \in \text{Row}(A)$. If $ax \leq x'$ for $x, x' \in \text{Row}(A)$ and $a \in S$ then $\overline{x'}a \leq \overline{x}$, and as such $(\phi x')a \leq \phi x$. Hence ϕ is antitone. Similarly the function $\psi : \text{Col}(A) \rightarrow \text{Row}(A)$ given by $y \mapsto \overline{y}A$ is antitone, so it remains to show that ψ is the inverse of ϕ . Let $uA \in \text{Row}(A)$. Then

$$\psi\phi(uA) = \overline{A\overline{uA}}A \leq uA \quad (6.14)$$

by (6.13) because $A\overline{uA} \leq A\overline{uA}$. We also have $uA \leq uA$, so by (6.13) again $u \leq \overline{A\overline{uA}}$. Therefore

$$uA \leq \overline{A\overline{uA}}A = \psi\phi(uA), \quad (6.15)$$

and as such $\psi\phi(uA) = uA$. Similarly $\phi\psi(Av) = Av$ for all $Av \in \text{Col}(A)$, so ψ is the inverse of ϕ . \square

In [8] it is proved that the row and column spaces of matrices with entries in \mathbb{FT} are anti-isomorphic. The notion of anti-isomorphism used in [8] is, in general, weaker than the present one, but in the case of \mathbb{FT} the two are equivalent. In [3] it is proved that if \mathcal{K} is a complete idempotent semiring with an anti-isomorphism $\mathcal{K} \rightarrow \mathcal{K}$ then the row and column spaces of any matrix with entries in \mathcal{K} are anti-isomorphic as complete lattices only, not (as in Theorem 6.5) as modules. This is because the anti-isomorphism $\mathcal{K} \rightarrow \mathcal{K}$ need not be an involution.

The completion of \mathbb{FT} is obtained from \mathbb{FT} by adjoining two new elements $-\infty$ and ∞ with $-\infty < a < \infty$ for all $a \in \mathbb{FT}$, setting $(-\infty)a = a(-\infty) = -\infty$ for all $a \in \mathbb{FT} \cup \{-\infty, \infty\}$ and $(\infty)a = (\infty)a = \infty$ for all $a \in \mathbb{FT} \cup \{-\infty, \infty\}$. The row and column spaces of matrices with entries in this semiring are known to be anti-isomorphic in the respective senses above [3, 8], but we conclude that they are in fact anti-isomorphic in the strongest sense because the completion of \mathbb{FT} is anti-involutive.

Theorem 6.6. *If S is an anti-involutive semiring then S is exact.*

Proof. Let $A \in S^{m \times n}$ and $B \in S^{p \times n}$ with $\text{Ker Row}(A) \subseteq \text{Ker Row}(B)$. We will show that $\text{Row}(B) \subseteq \text{Row}(A)$. Let $uB \in \text{Row}(B)$ and for convenience let $v = \overline{uB}$. Then $\overline{AvA} \leq \overline{v} = uB$ by (6.13) because $Av \leq Av$. As in the proof of Theorem 6.5, (6.13) gives $\overline{AvA} = Av$, so

$$(\overline{AvA}, v) \in \text{Ker Row}(A) \subseteq \text{Ker Row}(B). \quad (6.16)$$

Therefore $\overline{B\overline{AvA}} = Bv \leq \overline{u}$, and as such $uB \leq \overline{AvA}$ by (6.13) again. Hence $uB = \overline{AvA} \in \text{Row}(A)$, as required for $\text{Row}(B) \subseteq \text{Row}(A)$. Therefore (F1) and, by a similar argument, (F2) hold for all $A \in S^{m \times n}$. Thus S is exact by Theorem 3.2. \square

7. GREEN'S RELATIONS

The equivalence relations of Green are usually used to reveal the structure of semigroups [7, 9], but they can also be defined for (not necessarily square) matrices. Two matrices $A \in S^{m \times n}$ and $B \in S^{p \times n}$ are \mathcal{L} -related, written $A \mathcal{L} B$, if there are $M \in S^{p \times m}$ and $P \in S^{m \times p}$ with $MA = B$ and $PB = A$. We notice that $A \mathcal{L} B$ if and only if $\text{Row}(A) = \text{Row}(B)$. Similarly $A \in S^{m \times n}$ and $B \in S^{m \times q}$ are \mathcal{R} -related, written $A \mathcal{R} B$, if there are $N \in S^{n \times q}$ and $Q \in S^{q \times n}$ with $AN = B$ and $BQ = A$. Again, we notice that $A \mathcal{R} B$ if and only if $\text{Col}(A) = \text{Col}(B)$.

Green's \mathcal{D} relation is the relational composition of \mathcal{L} and \mathcal{R} . That is, $A \in S^{m \times n}$ and $B \in S^{p \times q}$ are \mathcal{D} -related if there is some $C \in S^{p \times n}$ with $A \mathcal{L} C \mathcal{R} B$.

Theorem 7.1. *Let S be an exact semiring, $A \in S^{m \times n}$ and $B \in S^{p \times q}$. Then the following conditions are equivalent.*

- (1) $\text{Row}(A) \cong \text{Row}(B)$ as left S -modules.
- (2) $A \mathcal{D} B$.

(3) $\text{Col}(A) \cong \text{Col}(B)$ as right S -modules.

Proof. We will only show the equivalence of conditions (2) and (3), the equivalence of (1) and (2) being dual.

If $A \mathcal{D} B$ then there is some $C \in S^{p \times n}$ with $A \mathcal{L} C \mathcal{R} B$. Therefore there are $M \in S^{p \times m}$ and $P \in S^{m \times p}$ with $MA = C$ and $PC = A$. There are also $N \in S^{n \times q}$ and $Q \in S^{q \times n}$ with $CN = B$ and $BQ = C$. The right linear function $\text{Col}(A) \rightarrow \text{Col}(B)$ given by $Av \mapsto MAv = BQv$ has inverse $\text{Col}(B) \rightarrow \text{Col}(A)$ given by $Bv \mapsto PBv = ANv$ because $PMA = A$ and $MPB = B$. Hence $\text{Col}(A) \cong \text{Col}(B)$.

Now let $\phi: \text{Col}(A) \rightarrow \text{Col}(B)$ be an isomorphism of right S -modules. For each $1 \leq i \leq p$ the composition of ϕ with projection to the i th entry yields a right linear function $\phi_i \in \text{Col}(A)^*$. By (G1), Theorem 3.3, each ϕ_i is given by $Av \mapsto u_i Av$ for some $u_i \in S^{1 \times m}$. Therefore ϕ is given by $Av \mapsto MAv$ with $\text{Col}(MA) = \text{Col}(B)$, where

$$M = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \in S^{p \times m}. \quad (7.1)$$

A similar argument shows that ϕ^{-1} is given by $Bv \mapsto PBv$ for some $P \in S^{m \times p}$ with $PMA = A$. Therefore $A \mathcal{L} MA \mathcal{R} B$, and as such $A \mathcal{D} B$. \square

Two matrices $A \in S^{m \times n}$ and $B \in S^{p \times q}$ are \mathcal{J} -related if and only if there are matrices $M \in S^{p \times m}$, $N \in S^{n \times q}$ with $MAN = B$ and matrices $P \in S^{m \times p}$, $Q \in S^{q \times n}$ with $PBQ = A$. It is clear from the definitions that $\mathcal{D} \subseteq \mathcal{J}$, but even if S is exact it need not be the case that $\mathcal{D} = \mathcal{J}$. For instance, it is known that $\mathcal{D} = \mathcal{J}$ for matrices with entries in $\mathbb{F}\mathbb{T}$ and that $\mathcal{D} \neq \mathcal{J}$ for matrices with entries in its completion [10]. It is also known that $\mathcal{D} = \mathcal{J}$ for matrices with entries in any Artinian ring that has a maximal ideal [2].

8. REMARKS AND OPEN QUESTIONS

In Section 4 we showed that proper quotients of principal ideal domains are exact rings. As remarked there, this result also follows from our characterisation of exactness as a weak form of self-injectivity (Theorem 3.4), since proper quotients of principal ideal domains are self-injective rings. In fact, all known (to the authors at least) examples of exact rings are also self-injective. This leaves open the question of whether there exist any exact rings that are not self-injective.

Corollary 4.5 not only establishes the exactness of proper quotients of principal ideal domains, it also provides a description of the orthogonal complement of the row space of a matrix. However, as with the relationship between row and column spaces (isomorphism in Section 4 vs. anti-isomorphism in Section 6), it seems unlikely that a general description of the kernels of matrices is obtainable using exactness alone. A natural extension of the present work would therefore be to determine the structure of the kernels of matrices with entries in an anti-involutive semiring.

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